TOPIC 3: Real Numbers

Real numbers can be represented by points on a number line. Each real number corresponds to a point on the number line, and each point on the line corresponds to a real number. Some important subsets of real numbers are: ***integers***, which are the numbers in the set $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$; ***fractions*** (also called ***rational numbers***), which are numbers of the form $\frac{a}{b}$, where a and b are integers and b ? 0; and ***irrational numbers***, which are numbers that cannot be written as fractions.

A ***square root*** of a real number a, written $\sqrt{a}$, is a real number n whose square is a; that is, $\sqrt{a} = n$ when $n^2 = a$.

Example 1: There are two square roots of 16, 4 and – 4, because $4^2 = 16$ and $(-4)^2 = 16$. Four is the *positive* square root of 16, written $\sqrt{16} = 4$, and – 4 is the *negative* square root of 16, written $-\sqrt{16} = -4$.

Recall that the square of any real number is either zero or positive. Therefore only nonnegative real numbers have square roots; negative numbers do not have real-number square roots. For example, $\sqrt{-16}$ is not a real number because there is no real number whose square is –16.

A ***perfect square*** is a number in the set {0, 1, 4, 9, 16, 25, …}. We can use perfect squares to approximate square roots.

Example 2: Approximate $\sqrt{17}$ without using a calculator.

Because 17 is between the perfect squares 16 and 25, $\sqrt{17}$ is between $\sqrt{16}$ and $\sqrt{25}$. Therefore, $\sqrt{17}$ is between 4 and 5.

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1. List the perfect squares less than 150.

For exercises 2-5, use perfect squares to approximate the square root without using a calculator.

2. $\sqrt{7}$ 3. $\sqrt{31}$ 4. $\sqrt{85}$ 5. $\sqrt{128}$

The ***absolute value*** of a real number a, written $|a|$, is the distance from 0 to a on the number line. The absolute value of a real number is never negative (Why?).

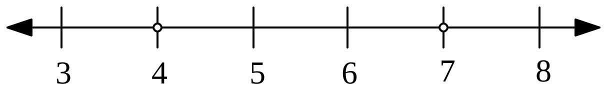
For exercises 6-11, find the absolute value.

6. $|-5|$ 7. $|12|$ 8. $|-12|$

9. $-|-5|$ 10. $|2-5|$ 11. $|5-2|$

Example 3: Let $a = 4$ and $b = 7$. Plot a and b on a number line, find the distance between a and b, and compute $|a-b|$.

The distance between 4 and 7 is 3 units (see the number line below) and $|a-b| = |4-7| = |-3| = 3$.



For exercises 12-15, plot a and b on a number line, find the distance between a and b, and compute $|a-b|$.

12. $a = 6$ and $b = 5$
13. $a = -3$ and $b = 8$
14. $a = -5$ and $b = -2$

15. If a and b are any real numbers, what does $|a-b|$ represent on the number line?

Every real number has a ***decimal representation***. For example, the decimal representation of $\frac{1}{2}$ is 0.5, which is a finite (terminating) decimal. Some real numbers, however, have an infinite (non-terminating) decimal representation. For instance, $\frac{2}{3} = 0.666\ldots$. We sometimes write 0.666 … as $0.\overline{6}$ to indicate the repeating pattern.

For exercises 16-19, find the decimal representation of the fraction.

16. $\frac{2}{5}$
17. $\frac{22}{8}$
18. $\frac{5}{3}$
19. $\frac{9}{11}$

Example 4: Write 0.56 as a fraction in reduced form.

Since 0.56 is 56 hundredths, we have $0.56 = \frac{56}{100} = \frac{23}{50}$.

For exercises 20-23, write the decimal as a fraction in reduced form.

20. 0.3
21. 0.68
22. 0.214
23. 0.1075

24. Can every terminating decimal be written as a fraction? Explain.

Example 5: Write $0.\overline{3}$ as a fraction.

Let $n = 0.\overline{3} = 0.333\ldots$. This is an infinite decimal where *one* digit is repeating. If we multiply n by 10, we move the decimal point *one* place to the right: $10n = 10(0.333\ldots) = 3.333\ldots$. Observe that the decimal parts of n and $10n$ are identical, so we obtain:

$$\begin{aligned} 10n - n &= 3.333\ldots - 0.333\ldots \\ 9n &= 3 \\ n &= \frac{3}{9} = \frac{1}{3} \end{aligned}$$

Example 6: Write $0.\overline{25}$ as a fraction.

We use the same method as above, but this time, since there are *two* digits that are repeating, we multiply by 100 to move the decimal point *two* places to the right.

Let $n = 0.\overline{25} = 0.2525\ldots$. Then $100n = 100(0.2525\ldots) = 25.2525\ldots$, and we have:

$$\begin{aligned} 100n - n &= 25.2525\ldots - 0.2525\ldots \\ 99n &= 25 \\ n &= \frac{25}{99} \end{aligned}$$

For exercises 25-28, write the decimal as a fraction.

25. $0.\overline{4}$ 26. $0.\overline{15}$ 27. $0.\overline{378}$ 28. $0.08\overline{3}$

29. Can every infinite decimal with a repeating pattern be written as a fraction? Explain.

Some real numbers have an infinite decimal representation with no repeating pattern. These numbers cannot be written as fractions, so they are irrational numbers. The real numbers p and $\sqrt{2}$ are examples of irrational numbers. As we see in the following example, an irrational number can be approximated to any degree of accuracy by real numbers that have a finite decimal representation.

Example 7: Approximate the irrational number $\sqrt{17}$ to three decimal places.

We use a calculator to find an approximation of $\sqrt{17}$:

$$\sqrt{17} \approx 4.123105626$$

We round the decimal to three places and obtain:

$$\sqrt{17} \approx 4.123$$

Notice that $\sqrt{17} \neq 4.123$ because $(4.123)^2 = 16.999129 \neq 17$.

30a. Using your calculator, find $\sqrt{2}$ and write this number in your notebook.
b. Now square the number (press the $\boxed{x^2}$ key on your calculator) to verify that its square is 2.
c. Clear the calculator, and then enter the number written in your notebook from part a above. Square this number. What answer does the calculator give?
d. Compare the two answers from parts b) and c) above. What has occurred?

31a. Approximate $\sqrt{17}$ to six decimal places.
b. Does $\sqrt{17} = 4.123105626$? Explain.

32. Can the infinite decimal $0.010010001000010000010000001\ldots$ be written as a fraction or is it an irrational number? Explain.