TOPIC 11: Lines

A ***linear equation in two variables*** is an equation that can be expressed in the form

$$ax + by = c$$

where a, b, and c are real numbers and a and b are not both zero. A ***solution*** of a linear equation in two variables is an ordered pair of numbers (x, y) that makes the equation true. For example, the ordered pair (1, 4) is a solution of the linear equation $4x + 2y = 12$, because $4(1) + 2(4) = 4 + 8 = 12$.

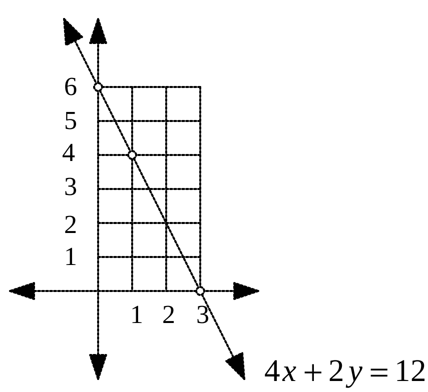
We can graph a linear equation in two variables by finding solutions of the equation and plotting the corresponding ordered pairs on a rectangular coordinate system. *It can be shown that the graph of a linear equation in two variables is always a straight line*. In this case we say there is a ***linear relationship*** between x and y.

Example 1: Graph $4x + 2y = 12$.

We already know that (1, 4) is a solution of the equation. To find other solutions we can choose values for one variable and find the corresponding values of the other. For example, if $x = 0$ then:

$$\begin{aligned} 4x + 2y &= 12 \\ 4 \cdot 0 + 2y &= 12 \\ 2y &= 12 \\ y &= 6 \end{aligned}$$

This shows that (0, 6) is a solution of the equation (Why?). In a similar manner we can show (3, 0) is a solution. The graph of $4x + 2y = 12$ is shown below.



We can see that each solution of the equation $4x + 2y = 12$ determines a point on the line. Conversely, each point on the line represents a solution of the equation.

Notice that the graph of $4x + 2y = 12$ crosses the y-axis at the point (0, 6). This point is the ***y-intercept*** of the graph. Similarly, the point (3, 0) is the ***x-intercept*** of the graph.

If we solve the equation $4x + 2y = 12$ for y, we obtain $y = -2x + 6$. We can now find solutions by choosing values for x and finding the corresponding values of y. A convenient way to record solutions of the equation $y = -2x + 6$ is to make a ***table of values***:

x	0	1	2	3
$y = -2x + 6$	6	4	2	0

The ordered pair (2, 2) is a solution of $y = -2x + 6$ because if x = 2 then $y = -2(2) + 6 = -4 + 6 = 2$.

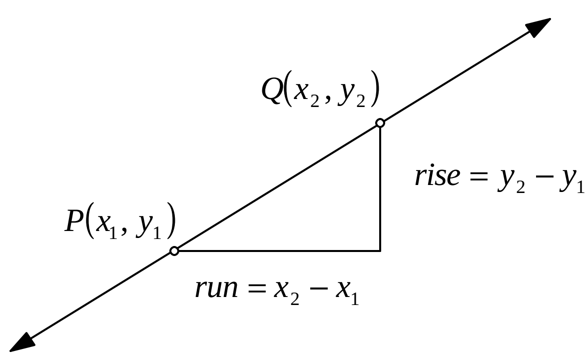
For exercises 1-6, make a table of values, graph the linear equation, and find the intercepts.

1. $2x + 8y = 24$
2. $x - 3y = 6$
3. $y = 4x$
4. $x + y = -5$
5. $y = \frac{5}{2}x + 1$
6. $x + 3y = 0$

One important feature of a line is its slope. The ***slope*** m of a line containing the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ is given by

$$m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope of a line is the ratio of the change in y (called the *rise*) to the change in x (called the *run*) between *any* two points P and Q on the line. See the figure below.



You can think of slope as a number that measures the "steepness" of a line.

Example 2: Find the slope of the line $4x + 2y = 12$.

We can choose any two points on the line to find the slope. If we choose (0, 6) as the first point and (1, 4) as the second point, the slope is

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-6}{1-0} = \frac{-2}{1} = -2$$

Any pair of points on the line would give the same slope.

Another way to understand the slope of the line $4x + 2y = 12$ is to look at the table of values above for the equation $y = -2x + 6$. Notice that the y-values *decrease* by 2 when the x-values *increase* by 1, so the slope of the line is $m = \frac{\text{change in } y}{\text{change in } x} = \frac{-2}{1}$. *The slope of a line indicates the change in y when x increases by one*. Thus the slope of a line is the *rate* that y is changing with respect to x.

Example 3: Suppose you set the cruise control on your car to 55 mph. This means that you are traveling at a constant speed of 55 mph. Using the distance formula $d = r \cdot t$, we have $d = 55t$, where d represents the distance traveled in miles after t hours. What is the slope of the line $d = 55t$?

We make a table of values and determine the rate at which distance is changing with respect to time:

t (hours)	0	1	2	3
$d = 55t$ (miles)	0	55	110	165

We see that distance *increases* by 55 miles when time *increases* by 1 hour, so the slope $m = \frac{\text{change in } d}{\text{change in } t} = \frac{55 \text{ miles}}{1 \text{ hour}} = 55 \text{ mph}$.

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7. For the line $4x + 2y = 12$ in Example 2 above:
   a. Find the slope of the line if the first point is (1, 4) and the second point is (0, 6).
   b. Find the slope of the line if the first point is (1, 4) and the second point is (2, 2).

For exercises 8-13, find the slope of the line using the table of values created in exercises 1-6.

8. $2x + 8y = 24$
9. $x - 3y = 6$
10. $y = 4x$
11. $x + y = -5$
12. $y = \frac{5}{2}x + 1$
13. $x + 3y = 0$

14. The cost C, in dollars, of a one-day truck rental is given by the linear equation $C = 0.3d + 60$, where d is the distance driven in miles.
    a. Graph the line $y = 0.3x + 60$. Is the graph of $C = 0.3d + 60$ (put C on the y-axis and d on the x-axis) the same as the graph of $y = 0.3x + 60$? Explain.
    b. What is the slope of $C = 0.3d + 60$? Interpret your answer as a rate.
    c. What is the y-intercept of $C = 0.3d + 60$? Explain your answer in terms of cost and miles driven.
    d. If it costs $240 to rent the truck for one day, how far was it driven?

15. Suppose the recommended heart rate for a healthy person when exercising is given by the linear equation $y = 172 - 0.75x$, where y is the heart rate in beats per minute x is the age of the person in years.
    a. Graph the line $y = 172 - 0.75x$.
    b. What is the slope of the line? Interpret your answer as a rate.
    c. If a person's recommended heart rate is 142 beats per minute today, what heart rate would be recommended three years from today?

16. The equation of a ***horizontal line*** is given by $y = b$, where b is a real number.
    a. Find three points on the horizontal line $y = 5$ and graph the line.
    b. What is the slope of the line?
    c. Does every horizontal line have the same slope? Explain.

17. The equation of a ***vertical line*** is given by $x = a$, where a is a real number.
    a. Find three points on the vertical line $x = -2$ and graph the line.
    b. What is the slope of the line? Explain your answer.
    c. What can you conclude about the slope of any vertical line?

18. Use your graphing calculator to compare the graphs of $y = mx$ for $m = 0.5$, 1, 2, and 4. Start by graphing $y = 0.5x$. Press the [Y=] key and type in $0.5x$ (use the [X, T, ?, *n*] key to enter the variable x). Then press the [GRAPH] key. To graph the next equation, $y = x$, press the [Y=] key, scroll down to the next line (the one that begins with Y_2=), and proceed as above. Continue until you have graphed all four equations. Which line rises most rapidly? Which line has the greatest slope?

19. Use your graphing calculator to compare the graphs of $y = mx$ for $m = -0.5, -1, -2$, and -4. Which line falls most rapidly? Which line has the smallest slope?