TOPIC 17: Solving Quadratic Equations

A ***quadratic equation*** is an equation that can be expressed in the form

$$ax^2 + bx + c = 0$$

where *a*, *b*, and *c* are real numbers and *a* ? 0. Note that a quadratic equation contains a squared term and no terms of higher degree (terms with an exponent greater than two).

There are several ways to solve quadratic equations. Some can be solved by factoring. To solve a quadratic equation by factoring we can use the following property.

Property of Zero Products
Let *a* and *b* be real numbers. Then $ab = 0$ if and only if $a = 0$ or $b = 0$ (or both).

This property says that if the product of two numbers is zero, then at least one of the numbers must be zero (and *vice versa*).

Example 1: Solve $(x+3)(x+2)=0$.

The equation $(x+3)(x+2)=0$ shows that the product of $x + 3$ and $x - 2$ is zero. In order for the product to be zero, at least one of the factors must be zero. Therefore,

$x + 3 = 0$ or $x + 2 = 0$ [property of zero products]
$x = -3$ or $x = -2$

For exercises 1-3, use the zero products property to solve the equation.

1. $(x-5)(x+2)=0$
2. $\left(\frac{1}{2}x+3\right)(2x-3)=0$
3. $x(5x+1)=0$

The equation $(x+3)(x+2)=0$ above does not appear to be a quadratic equation. However, if we multiply the two expressions (factors) on the left side of the equation we get the quadratic equation $x^2+5x+6=0$. To solve a quadratic equation by ***factoring*** we reverse this process. We first write the equation as a product of two expressions (factors) equal to zero and then use the property of zero products. For example, to solve the quadratic equation $x^2+5x+6=0$ by factoring, we first write it as $(x+3)(x+2)=0$ and then use property of zero products. The equation $(x+3)(x+2)=0$ is the factored form of $x^2+5x+6=0$.

Example 2: Solve $x^2 - 5x = 6$ by factoring.

We must write the equation in the form $ax^2 + bx + c = 0$, so we start by subtracting 6 from both sides.

$$x^2 - 5x = 6$$
$$x^2 - 5x - 6 = 0 \quad \text{[subtracting 6 from both sides]}$$
$$(x-6)(x+1) = 0 \quad \text{[factoring]}$$
$$x - 6 = 0 \text{ or } x + 1 = 0 \quad \text{[property of zero products]}$$
$$x = 6 \text{ or } x = -1$$

Example 3: Solve $4x^2 + 4x - 15 = 0$ by factoring.

$$4x^2 + 4x - 15 = 0$$
$$(2x-3)(2x+5) = 0 \quad \text{[factoring]}$$
$$2x - 3 = 0 \text{ or } 2x + 5 = 0 \quad \text{[property of zero products]}$$
$$x = \frac{3}{2} \text{ or } x = -\frac{5}{2}$$

When solving quadratic equations by factoring it is sometimes helpful to remember some special cases:

$$A^2 + 2AB + B^2 = (A+B)^2$$
$$A^2 - 2AB + B^2 = (A-B)^2$$
$$A^2 - B^2 = (A+B)(A-B)$$

For exercises 4-12, solve the quadratic equation by factoring.

4. $x^2 - 2x - 15 = 0$
5. $x^2 - 36 = 0$
6. $x^2 - 35 = -2x$
7. $x^2 + 24x + 144 = 0$
8. $2x^2 + 5x = 12$
9. $4x^2 - 1 = 0$
10. $6x^2 - 19x + 10 = 0$
11. $4x^2 - 12x + 9 = 0$
12. $(x+1)(x-2) = 4$

For exercises 13-17, solve the word problem by writing a quadratic equation and then factoring. Write your answer in a complete sentence.

13. The length of a rectangular garden is 4 meters greater than the width. The area of the garden is 96 square meters. Find the length and the width of the garden.

14. The base of a certain triangle is 10 cm longer than its height. The area of the triangle is 28 cm^2. Find the base and the height of the triangle.

15. Engine failure forced Susan to pilot her plane to an emergency landing. To land, Susan's plane glided 17,000 ft over a 15,000-ft stretch of deserted highway. From what altitude did the descent begin?

16. The product of two consecutive odd integers is 255. Find the integers.

17. Eight times a number is 15 more than the square of that number. Find all such numbers.

Some quadratic equations cannot be solved by factoring. For example, $x^2 + 4x - 6 = 0$ is one such equation. We can use the method of ***completing the square*** (explained below) and the following property to solve *any* quadratic equation.

Property of Square Roots
For any nonnegative real number k, if $x^2 = k$ then $x = \pm\sqrt{k}$.

Example 4: Solve $x^2 + 4x - 6 = 0$ by the method of completing the square.

Step 1. We start the method of completing the square by writing the equation with the constant term on the right side.

$$x^2 + 4x - 6 = 0$$
$$x^2 + 4x = 6 \qquad \text{[adding 6 to both sides]}$$

Step 2. We now take one-half the coefficient of x, square it, and add the result to both sides of the equation. The coefficient of x is 4, so we add $\left(\frac{4}{2}\right)^2 = 2^2 = 4$ to both sides.

$$x^2 + 4x = 6$$
$$x^2 + 4x + 4 = 6 + 4 \qquad \text{[adding 4 to both sides]}$$
$$x^2 + 4x + 4 = 10$$

Step 3. We factor the left side and apply the property of square roots.

$$x^2 + 4x + 4 = 10$$
$$(x + 2)^2 = 10 \qquad \text{[factoring]}$$
$$x + 2 = \pm\sqrt{10} \qquad \text{[property of square roots]}$$
$$x = -2 \pm \sqrt{10} \qquad \text{[subtracting 2 from both sides]}$$

The quadratic equation $x^2 + 4x - 6 = 0$ has two solutions, $x = -2 + \sqrt{10}$ and $x = -2 - \sqrt{10}$.

In summary, to solve the quadratic equation $x^2 + bx + c = 0$ by completing the square:

1. Write the equation with the constant term on the right side.
2. Take one-half the coefficient of *x*, square it, and add the result to both sides.
3. Factor the expression on the left side and apply the property of square roots.

We can extend the method of completing the square to *any* quadratic equation, $ax^2 + bx + c = 0$, where $a \neq 0$, by first dividing both sides of the equation by *a* and then following the three steps given above.

For exercises 18-20, solve the quadratic equation by completing the square.

18. $x^2 + 6x + 4 = 0$ 19. $x^2 - 8x - 7 = 0$ 20. $4x^2 - 6x + 1 = 0$

By completing the square on $ax^2 + bx + c = 0$, where $a \neq 0$, we can establish a general formula for the solutions of a quadratic equation. This formula is known as the ***quadratic formula*** and the solutions are given by:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The two *possible* solutions for a quadratic equation are:

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

However, not every quadratic equation has two real-number solutions. Some only have one solution, and others have no real solutions (the solutions are complex numbers).

For exercises 21-26, solve the equation using the quadratic formula. If there are no real solutions, say so.

21. $x^2 - 8x + 16 = 0$ 22. $-2x^2 - 37x - 93 = 0$ 23. $x^2 + x + 1 = 0$

24. $3x^2 = 12 - 5x$ 25. $4x^2 - 20x + 21 = 0$ 26. $8 - 3x - x^2 = 0$

27. The expression $b^2 - 4ac$ is the ***discriminant*** of the quadratic equation $ax^2 + bx + c = 0$. If a quadratic equation has no real solutions, what is the value of the discriminant? What if there is only one real solution? Two real solutions?

28. Derive the quadratic formula by completing the square on $ax^2 + bx + c = 0$.