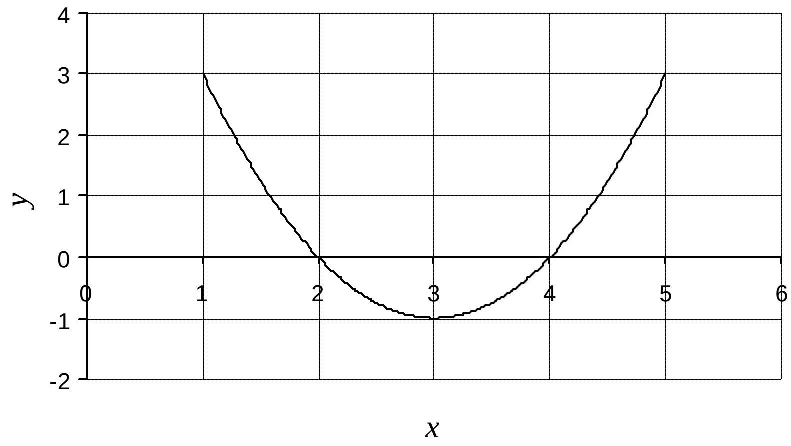
TOPIC 18: Graphing $y = ax^2 + bx + c$

The equation $y = ax^2 + bx + c$, where *a* ? 0, represents a ***quadratic function***. We can easily graph a quadratic function with our graphing calculator. For example, a portion of the graph of $y = x^2 - 6x + 8$ is shown below:



The graph of $y = x^2 - 6x + 8$ crosses the *x*-axis at the points (2, 0) and (4, 0). These are the ***x-intercepts*** of the graph. Notice that the *y*-coordinate of the *x*-intercepts is zero. Therefore we can find the *x*-intercepts *without graphing* by letting y = 0 and then solving the resulting equation for *x*:

$$y = x^2 - 6x + 8$$
$$0 = x^2 - 6x + 8$$

The last equation is a quadratic equation, which we can solve by factoring (or by the quadratic formula):

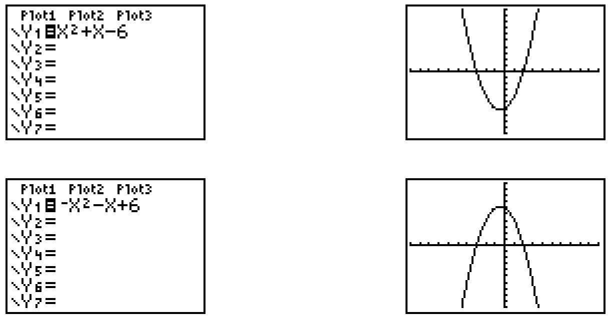
$$0 = x^2 - 6x + 8$$
$$0 = (x - 2)(x - 4)$$
$$x = 2, 4$$

The solutions are sometimes referred to as the ***roots*** of the equation. In general, the roots of the quadratic equation $ax^2 + bx + c = 0$ are the *x*-coordinates of the *x*-intercepts of the graph of $y = ax^2 + bx + c$.

For exercises 1-6, find the *x*-intercepts without graphing. To verify your answers, graph the quadratic function using your graphing calculator.

1. $y = x^2 - 3x - 4$
2. $y = x^2 - 2x - 2$
3. $y = x^2 - 4x + 5$
4. $y = x^2 + 10x + 25$
5. $y = 2x^2 - 7x + 3$
6. $y = 4x^2 + 3x - 1$

The graph of $y = ax^2 + bx + c$ is a ***parabola***. Parabolas either open upward or open downward depending on the value of a. If $a > 0$ then the parabola ***opens upward***, and if $a < 0$ then it ***opens downward***. The following graphing calculator screens illustrate this idea.



We see that the graph of $y = x^2 + x - 6$ $(a = 1)$ opens upward and the graph of $y = -x^2 - x + 6$ $(a = -1)$ opens downward.

If a parabola opens upward, the lowest point on the graph is the ***vertex***. For example, the vertex of the graph of $y = x^2 - 6x + 8$ above is the point (3, –1). Similarly, the vertex of a parabola that opens downward is the highest point on the graph.

We know that the x-intercepts of the graph of $y = x^2 - 6x + 8$ are the *distinct* points (2, 0) and (4, 0), where 2 and 4 are the roots of $x^2 - 6x + 8 = 0$. Notice that the point (3, 0) is the *midpoint* of the line segment with endpoints (2, 0) and (4, 0), and it has the same x-coordinate as the vertex (3, –1). Since the x-coordinate of the midpoint is the average of the x-coordinates of the endpoints, *the x-coordinate of the vertex is the average of the roots*:

$$x = \frac{2+4}{2} = 3$$

To find the y-coordinate of the vertex we substitute $x = 3$ into $y = x^2 - 6x + 8$:

$$y = x^2 - 6x + 8$$
$$y = 3^2 - 6(3) + 8$$
$$y = 9 - 18 + 8$$
$$y = -1$$

For exercises 7-9, find the vertex without graphing. Then verify your answers by graphing the quadratic function.

7. $y = x^2 - 9$ 8. $y = x^2 - 2x - 24$ 9. $y = -2x^2 + 3x - 1$

10. Suppose the graph of $y = ax^2 + bx + c$ has *distinct* x-intercepts $(x_1, 0)$ and $(x_2, 0)$.
    a. Use the quadratic formula to find x_1 and x_2.
    b. Let x be the average of x_1 and x_2. Find x. This is the x-coordinate of the vertex.
    c. Find the y-coordinate of the vertex.

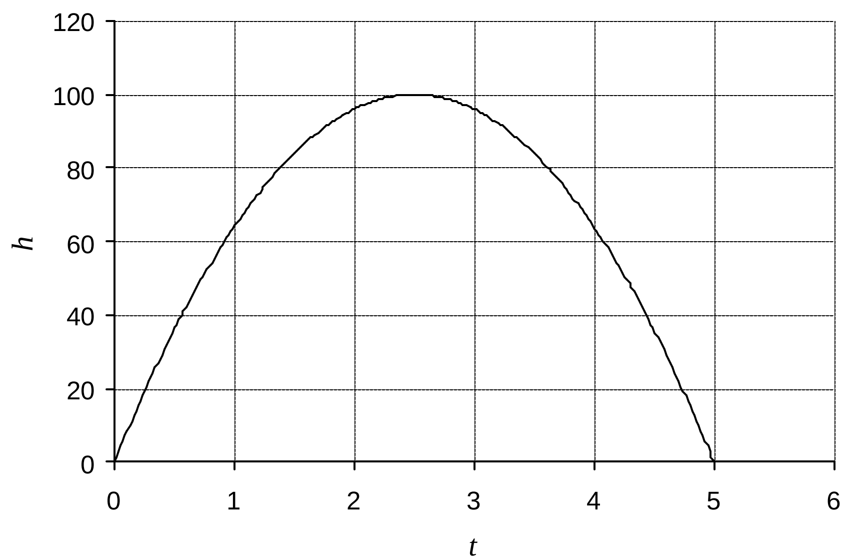
11. If the graph of $y = ax^2 + bx + c$ does *not* have two distinct x-intercepts, can we use the result of exercise 10b to find the x-coordinate of the vertex? To answer this question, graph the quadratic functions below and find the x-intercepts. Does the formula $x = -\dfrac{b}{2a}$ appear to give the x-coordinate of the vertex?

    a. $y = x^2 + 4x + 4.$ b. $y = -x^2 + 4x - 5.$

A common problem in mathematics involves finding the maximum or minimum value of a quantity. As we see in the following example, if the quantity is represented by a quadratic function, we can solve the problem by finding the coordinates of the vertex. We will use the fact that $x = -\dfrac{b}{2a}$ is the x-coordinate of the vertex of the parabola with equation $y = ax^2 + bx + c$.

Example 1: When an object is shot upward with an initial velocity of 80 f/s, its height h (in feet) after t seconds is given by the quadratic function $h = -16t^2 + 80t$. How long will it take the object to reach its maximum height? What is the maximum height?

We begin by graphing $h = -16t^2 + 80t$:



Example 1: *(continued)* It appears from the graph that the vertex of the parabola is the point (2.5, 100). Recall that the vertex of a parabola that opens downward is the highest point on the graph. Therefore we can find the maximum height by finding the *h*-coordinate of the vertex. First we find the *t*-coordinate of the vertex:

$$t = -\frac{b}{2a} = -\frac{80}{2(-16)} = 2.5$$

Thus it takes the object 2.5 seconds to reach its maximum height. To find the maximum height we substitute $t = 2.5$ into $h = -16t^2 + 80t$:

$$h = -16t^2 + 80t$$
$$h = -16(2.5)^2 + 80(2.5)$$
$$h = -100 + 200$$
$$h = 100$$

The maximum height of the object is 100 feet.

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12. The Flying Kite Company has found that the daily profit *p* (in dollars) from selling x kites is $p = -x^2 + 500x - 52{,}500$. How many kites should the company sell per day to maximize its profit? What is the maximum profit?

13. Bill's Bicycle Shop has determined that when x hundred bicycles are built, the average cost per bicycle *C* (in hundreds of dollars) is $C = 0.1x^2 - 0.7x + 2.425$. How many bicycles should the shop build in order to minimize the average cost per bicycle? What is the minimum average cost per bicycle?

14. What is the minimum product of two numbers that differ by 5? What are the numbers?

15. Sally has 60 feet of fence to enclose a rectangular garden, using one side of her house as one side of the rectangle. What are the dimensions of the rectangle that produce the maximum area? What is the maximum area?